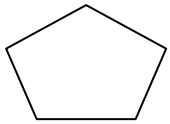
| | |
|---|---|
| Name of Course | : **CBCS B.Sc. (H) Mathematics** |
| Unique Paper Code | : **32351302_OC** |
| Name of Paper | : **Core-Group Theory I** |
| Semester | : **III** |
| Duration | : **3 hours** |
| Maximum Marks | : **75 Marks** |

*Attempt any four questions. All questions carry equal marks.*

1. Describe all symmetries of the following pictures.

   (a) 

   (b) 

   Write Cayley's table for each of the corresponding two symmetric groups. Further, find order of each element in the two symmetric groups. If F and R represent a fixed reflection and a fixed rotation respectively in any of the two groups, then find $FR^7FR^2F$ . Find the center of each of the two groups. Explain whether the groups are abelian.

2. Show that the set of all elements of finite order in an abelian group is a subgroup of the group. Consider the following group under matrix multiplication.

   $GL(2,\mathbb{Z}_2) = \{A | A \text{ is a } 2\times 2 \text{ matrix with entries from } \mathbb{Z}_2 \text{ and determinant } |A| \neq 0\}$

   Write $|GL(2,\mathbb{Z}_2)|$. Let

   $H = \{A \in GL(2,\mathbb{Z}_2) | A^t = A, \text{where } A^t \text{ denotes the transpose matrix of } A\}$

   Is H a subgroup of GL(2,$\mathbb{Z}_2$) ? Justify! Compute order of the element $\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$ in $GL(2,\mathbb{Z}_2)$. Explain without actual calculation why there is no element in $GL(2,\mathbb{Z}_2)$ of order 6.

3. Let $S_4$ and $S_5$ denote symmetric groups of degree 4 and 5 respectively. What is the order of the permutation $\alpha = (1245)(314)$ in $S_5$? Find $\alpha^{-1}$. Is the permutation $\alpha$ even? Give an element $\beta$ of order 4 in $S_4$. Write all the elements of the cyclic group generated by $\beta$. Let $H = \{\alpha \epsilon S_4 : \alpha(2) = 2\}$ and $K = \{\alpha \epsilon S_4 : \alpha(4) = 4\}$. Prove that H is isomorphic to K.

4. Let $\mathbb{Z}$ be the additive group of integers. Let $H = \{0, \pm 5, \pm 10, \pm 15, \ldots\}$. Find all the left cosets of H in $\mathbb{Z}$. What is the index of H in $\mathbb{Z}$? Find all the subgroups of $\mathbb{Z}_{15}$ the

group of integers under addition modulo 15. Also write down all their generators. Show that U(25) is a cyclic group under multiplication modulo 25. Find all its generators.

5. If G is a group, N a normal subgroup of G and H any subgroup of G, show that
   (i) NH is a subgroup of G,
   (ii) HN is a subgroup of G,
   (iii) N is a normal subgroup of HN.
   If G = GL (2,$\mathbb{R}$) and H = $\left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R} \text{ and } xy \neq 0\right\}$, check whether H is a normal subgroup of G.

6. Find all homomorphism mappings from $\mathbb{Z}_{10}$ to $\mathbb{Z}_4$. If $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is defined as $\varphi(x) = 9x \quad \forall x \in \mathbb{Z}_{12}$, prove that $\varphi$ is a homomorphism. Find its kernel.